Discrete Mathematics 310 (2010) 1456–1462

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Packing edge-disjoint cycles in graphs and the cyclomatic number

Jochen Harant [a], Dieter Rautenbach [a,*], Peter Recht [b], Friedrich Regen [a]

[a] Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany

[b] Lehrstuhl für Operations Research und Wirtschaftsinformatik, Universität Dortmund, D-44227 Dortmund, Germany

ARTICLE INFO

Article history:
Received 7 October 2008
Received in revised form 14 July 2009
Accepted 17 July 2009
Available online 30 July 2009

Keywords:
Graph
Cycle
Packing
Cyclomatic number

ABSTRACT

For a graph G let $\mu(G)$ denote the cyclomatic number and let $\nu(G)$ denote the maximum number of edge-disjoint cycles of G.

We prove that for every $k \geq 0$ there is a finite set $\mathcal{P}(k)$ such that every 2-connected graph G for which $\mu(G) - \nu(G) = k$ arises by applying a simple extension rule to a graph in $\mathcal{P}(k)$. Furthermore, we determine $\mathcal{P}(k)$ for $k \leq 2$ exactly.

1. Introduction

We consider finite and undirected graphs $G = (V_G, E_G)$ with vertex set V_G and edge set E_G which may contain multiple edges but no loops. We use standard terminology [10] and only recall some basic notions. If an edge $e \in E_G$ has the two incident vertices u and v in V_G, then we write $e = uv$. The degree $d_G(u)$ in G of a vertex $u \in V_G$ is the number of edges $e \in E_G$ incident with u. A path in G of length $l \geq 0$ is a sequence $v_0e_1v_1e_2 \dots e_lv_l$ of distinct vertices $v_0, v_1, \dots, v_l \in V_G$ and distinct edges $e_i = v_{i-1}v_i \in E_G$ for $1 \leq i \leq l$. A cycle in G of length $l \geq 2$ is a sequence $v_1e_2v_2 \dots e_lv_le_1v_1$ such that $v_1e_2v_2 \dots e_lv_l$ is a path of length $(l-1)$ and $e_1 = v_lv_1 \in E_G$. The subgraph induced by some set $U \subseteq V_G$ is denoted by $G[U]$. An *ear* of G is a path in G of length at least 1 such that all its internal vertices have degree 2 in G. An ear of G is *maximal*, if it is not properly contained in another ear of G. If P is an ear of G and I is the set of internal vertices of P, then we say that G arises from $G' = (V_G \setminus I, E_G \setminus E_P)$ by *adding the ear* P and that G' arises from G by *removing the ear* P. Whitney [10,14] proved that a graph of order at least 2 is 2-connected if and only if it has an *ear decomposition*, i.e. it arises from a chordless cycle by iteratively adding ears. A graph is a *cactus graph*, if all of its cycles are edge-disjoint which is equivalent to the fact that all of its blocks are cycles or edges.

The *cyclomatic number* of a graph G with $\kappa(G)$ components is

$$\mu(G) = |E_G| - |V_G| + \kappa(G).$$

A *cycle packing* $\mathcal{C}$ *of* G *of order* l is a set of l edge-disjoint cycles of G. The maximum order of a cycle packing of G is denoted by

$$\nu(G).$$

A cycle packing of maximum order is called *optimal*. For a cycle packing $\mathcal{C}$, the set of edges contained in some cycle in $\mathcal{C}$ is denoted by

$$E_{\mathcal{C}}.$$

* Corresponding author.
E-mail addresses: jochen.harant@tu-ilmenau.de (J. Harant), dieter.rautenbach@tu-ilmenau.de (D. Rautenbach), peter.recht@tu-dortmund.de (P. Recht), friedrich.regen@tu-ilmenau.de (F. Regen).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.07.017

Our research in the present paper is motivated by the well known inequality

$$\nu(G) \leq \mu(G)$$

which holds for every graph G. As our main result, we prove that for every fixed $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ there is a finite set $\mathcal{P}(k)$ of graphs such that every 2-connected graph G for which

$$\mu(G) - \nu(G) = k$$

arises by applying a simple extension rule to one of the graphs in $\mathcal{P}(k)$, i.e. there are essentially only finitely many configurations which cause $\mu(G)$ and $\nu(G)$ to deviate by k. Furthermore, we determine $\mathcal{P}(k)$ for $k \leq 2$ exactly.

The results which are most related to ours concern the minimum difference $p(k)$ between the size $|E_G|$ and the order $|V_G|$ of a graph G which forces the existence of k edge-disjoint cycles, i.e.

$$p(k) = \min\{p \mid \nu(G) \geq k \,\forall\, G = (V_G, E_G) \text{ with } |E_G| - |V_G| \geq p\}.$$

There are several classical results concerning this parameter

$$p(k) = \begin{cases} 0, & k = 1 \\ 4, & k = 2[6] \\ 10, & k = 3[8] \\ 18, & k = 4[1,12] \\ \Theta(k \log k) & [6,11,13,12]. \end{cases}$$

Recently, algorithmic aspects of cycle packing problems have received considerable attention. While the problem to determine optimal cycle packings is APX-hard [3,7] (see [4,9] for related results concerning packings of shortest cycles) and remains NP-hard even when restricted to Eulerian graphs of maximum degree 4 [2], there are simple approximation algorithms [3,7].

In Section 2 we prove our main result about the finiteness of $\mathcal{P}(k)$ and in Section 3 we determine $\mathcal{P}(k)$ for $k \leq 2$ exactly.

2. Graphs G with $\mu(G) - \nu(G) = k$

In this section we study the graphs G for which $\mu(G)$ and $\nu(G)$ differ by some fixed k. It is well known — and easy to see — that the graphs G with $\mu(G) - \nu(G) = 0$ are exactly the cactus graphs, i.e. their blocks are either edges or arise by possibly subdividing the edges of a cycle of length 2.

For $k \in \mathbb{N}_0$ let

$$\mathcal{G}(k)$$

denote the set of 2-connected graphs G with $\mu(G) - \nu(G) = k$. In view of the above remark about cactus graphs, we obtain that $G \in \mathcal{G}(0)$ if and only if G is 2-connected cactus graph which implies that G is a cycle or an edge. The next lemma implies that in order to characterize the graphs G with $\mu(G) - \nu(G) = k$, it suffices to characterize the 2-connected graphs with this property.

Lemma 1. *Let $k \in \mathbb{N}_0$. If G is a graph with $\mu(G) - \nu(G) = k$ whose blocks $B_1, B_2, \ldots, B_l$ satisfy $B_i \in \mathcal{G}(k_i)$ for $1 \leq i \leq l$, then $k = k_1 + k_2 + \cdots + k_l$.*

Proof. This follows immediately from the fact that every cycle of G is entirely contained in some block of G. □

In order to explain the simple extension rule mentioned in the introduction, we need some more notation.

Let $l \in \mathbb{N}_0$.

An *l-cycle-path* is a cactus with at most 2 endblocks and exactly l cycles.

An *l-cycle-path-subgraph* of a graph $G = (V_G, E_G)$ with *attachment vertices u and v* is an induced subgraph $H = (V_H, E_H)$ of G which is an l-cycle-path such that u and v are two distinct vertices of H for which $d_G(w) = d_H(w)$ for all $w \in V_H \setminus \{u, v\}$ and $H + uv = (V_H, E_H \cup \{uv\})$ is 2-connected, i.e. only the attachment vertices may have neighbours outside of V_H and, if H has more than one block, then the attachment vertices are two non-cutvertices from the two endblocks of H. Note that a 0-cycle-path-subgraph of G with attachment vertices u and v is an ear of G with endvertices u and v.

A graph $H = (V_H, E_H)$ is said to arise from a graph $G = (V_G, E_G)$ *by replacing the edge $e = uv \in E_G$ with an l-cycle-path*, if H has an l-cycle-path-subgraph $Q = (V_Q, E_Q)$ with attachment vertices u and v such that (cf. Fig. 1)

$$V_G = V_H \setminus (V_Q \setminus \{u, v\}) \text{ and}$$
$$E_G = (E_H \setminus E_Q) \cup \{e\}.$$

A graph H is said to *extend* a graph G, if there is an optimal cycle packing $\mathcal{C}$ of G such that H arises from G by replacing every edge $e \in E_{\mathcal{C}}$ with a 0-cycle-path and replacing every edge $e \in E_G \setminus E_{\mathcal{C}}$ with an l-cycle-path for some $l \in \mathbb{N}_0$. A graph H is said to be *reduced*, if there is no graph G different from H such that H extends G.

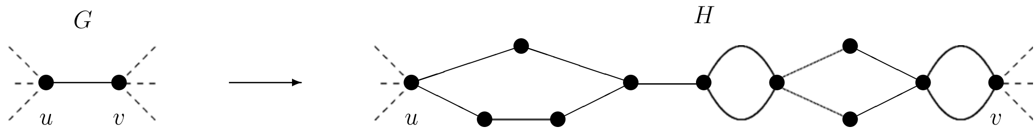

Fig. 1. Replacing the edge $e = uv \in E_G$ with a 4-cycle-path.

For $k \in \mathbb{N}_0$ let

$\mathcal{P}(k)$

denote the set of reduced graphs in $\mathcal{G}(k)$. Note that $\mathcal{P}(0)$ contains exactly two elements, an edge and a cycle of length 2. It is instructive to verify that for $k \geq 1$ a graph in $\mathcal{P}(k)$ contains neither vertices of degree at most 2 nor l-cycle-path-subgraphs for $l \geq 2$.

The next lemma summarizes some important properties of the above extension notion.

Lemma 2. *If $G_0 \in \mathcal{G}(k)$, G_1 extends G_0, and G_2 extends G_1, then*

(i) $G_1 \in \mathcal{G}(k)$,
(ii) G_2 *extends* G_0, *and*
(iii) *every graph in* $\mathcal{G}(k)$ *extends a graph in* $\mathcal{P}(k)$.

Proof. Let $\mathcal{C}_0$ be an optimal cycle packing of G_0 such that G_1 arises from G_0 by replacing every edge $e \in E_{G_0}$ with an l_e-cycle-path L_e with $l_e = 0$ for $e \in E_{\mathcal{C}_0}$. Let $\mathcal{C}'_1$ denote the set of the

$$\sum_{e \in E_{G_0}} l_e$$

edge-disjoint cycles contained in the l_e-cycle-paths L_e for $e \in E_{G_0}$.

Clearly,

$$\mu(G_1) = \mu(G_0) + |\mathcal{C}'_1|.$$

Since the set of cycles in G_1 which are subdivisions of the cycles in $\mathcal{C}_0$ together with the cycles in $\mathcal{C}'_1$ form a cycle packing of G_1, we obtain $\nu(G_1) \geq \nu(G_0) + |\mathcal{C}'_1|$.

Let $\mathcal{C}_1$ be an optimal cycle packing of G_1 such that G_2 arises from G_1 by replacing every edge $f \in E_{G_1}$ with an h_f-cycle-path H_f with $h_f = 0$ for $f \in E_{\mathcal{C}_1}$ and such that subject to this condition

$$|\mathcal{C}'_1 \cap \mathcal{C}_1|$$

is largest possible.

If E'_1 is an arbitrary set of edges which contains exactly one edge from every cycle in $\mathcal{C}'_1$, then removing the $|\mathcal{C}'_1|$ edges in E'_1 from G_1 can delete at most $|\mathcal{C}'_1|$ cycles in $\mathcal{C}_1$. Since the remaining cycles are subdivisions of cycles in G_0, we obtain $\nu(G_0) \geq \nu(G_1) - |\mathcal{C}'_1|$.

In view of the above, this implies that

$$\nu(G_1) = \nu(G_0) + |\mathcal{C}'_1| \tag{1}$$

and hence (i).

Furthermore, this implies that every edge contained in a cycle in $\mathcal{C}'_1$ belongs to $E_{\mathcal{C}_1}$ and edges contained in different cycles in $\mathcal{C}'_1$ are contained in different cycles in $\mathcal{C}_1$. (Otherwise there would be a choice for E'_1 such that removing the edges in E'_1 would only delete at most $|\mathcal{C}'_1| - 1$ cycles in $\mathcal{C}_1$, which implies the contradiction $\nu(G_0) \geq \nu(G_1) - |\mathcal{C}'_1| + 1$.)

If follows that, if $l_e \geq 2$ for some $e \in E_{G_0}$, then $\mathcal{C}_1$ necessarily contains the l_e edge-disjoint cycles contained in the l_e-cycle-path L_e.

Furthermore, if $l_e = 1$ for some $e \in E_{G_0}$ and $\mathcal{C}_1$ does not contain the unique cycle C_e contained in the 1-cycle-path L_e, then there are exactly two cycles C'_e and C''_e in $\mathcal{C}_1$ which contain E_{C_e}. Since $(E_{C'_e} \cup E_{C''_e}) \setminus E_{C_e}$ contains the edge set of a cycle C'''_e,

$$\tilde{\mathcal{C}}_1 = (\mathcal{C}_1 \setminus \{C'_e, C''_e\}) \cup \{C_e, C'''_e\}$$

is an optimal cycle packing of G_1 such that $E_{\tilde{\mathcal{C}}_1} \subseteq E_{\mathcal{C}_1}$ and

$$|\mathcal{C}'_1 \cap \tilde{\mathcal{C}}_1| > |\mathcal{C}'_1 \cap \mathcal{C}_1|$$

which is a contradiction to the choice of $\mathcal{C}_1$.

Hence $\mathcal{C}'_1 \subseteq \mathcal{C}_1$. By (1), the cycles in $\mathcal{C}_1 \setminus \mathcal{C}'_1$ are the subdivisions of the cycles in an optimal cycle packing $\mathcal{C}'_0$ of G_0. Clearly, $l_e > 0$ implies $e \notin E_{\mathcal{C}'_0}$. Since $h_f > 0$ for some $f \in E_{G_1} \setminus E_{\mathcal{C}_1}$ implies that f is a bridge of an l_e-cycle-path L_e with $e \notin E_{\mathcal{C}'_0}$, it follows that G_2 extends G_0, i.e. (ii) holds.

By definition, for every graph $H \in \mathcal{G}(k)$ there is a graph $G \in \mathcal{P}(k)$ such that H arises from G by a finite sequence of extensions. By (ii), H extends G and (iii) follows. This completes the proof. $\square$

We proceed to our main result.

Theorem 3. *The set $\mathcal{P}(k)$ is finite for every $k \in \mathbb{N}_0$.*

Proof. We will prove the result by induction on k.

Since $|\mathcal{P}(0)| = 2$, we may assume that $k \geq 1$.

We will argue that every graph in $\mathcal{P}(k)$ arises from some graph in $\mathcal{P}(k-1)$ by applying a subset of a finite set of operations. Since, by induction, $\mathcal{P}(k-1)$ is finite, this clearly implies that $\mathcal{P}(k)$ is finite.

Let $H \in \mathcal{P}(k)$.

Let $H_0, H_1, \ldots, H_t = H$ be an ear decomposition of H, i.e. H_0 is a cycle and, for $1 \leq i \leq t$, the graph H_i arises from H_{i-1} by adding an ear. Clearly, for $1 \leq i \leq t$, $\mu(H_i) = \mu(H_{i-1}) + 1$ and $\nu(H_{i-1}) \leq \nu(H_i) \leq \nu(H_{i-1}) + 1$ which implies that

$$\mu(H_{i-1}) - \nu(H_{i-1}) \leq \mu(H_i) - \nu(H_i) \leq \mu(H_{i-1}) - \nu(H_{i-1}) + 1.$$

Therefore, since $H_0 \in \mathcal{G}(0)$, $H = H_t \in \mathcal{G}(k)$ and $k \geq 1$, there is some $1 \leq i^* \leq t$ such that $H_{i^*-1} \in \mathcal{G}(k-1)$ and $H_i \in \mathcal{G}(k)$ for $i^* \leq i \leq l$. Setting $l = t - i^* + 1$ and $G_i = H_{i^*+i-1}$ for $0 \leq i \leq l$ yields a sequence of 2-connected graphs

$$G_0, G_1, \ldots, G_l$$

such that

- $G_l = H$,
- G_i arises by adding the ear P_i to G_{i-1} for $1 \leq i \leq l$,
- $\nu(G_0) = \nu(G_1)$ and
- $\nu(G_{i-1}) = \nu(G_i) - 1$ for $2 \leq i \leq l$.

We assume that the sequence is chosen to be shortest possible, i.e. l is minimum.

Note that $G_0 \in \mathcal{G}(k-1)$ and $G_i \in \mathcal{G}(k)$ for $1 \leq i \leq l$.

By Lemma 2 (iii), G_0 extends some graph

$$G \in \mathcal{P}(k-1).$$

Let

$$\mathcal{C}_l$$

be an optimal cycle packing of $H = G_l$.

Since for $l \geq 2$ we have $\nu(G_{l-1}) = \nu(G_l) - 1$ and removing the ear P_l from G_l can only affect one cycle from $\mathcal{C}_l$, the ear P_l is contained in a unique cycle

$$C_l \in \mathcal{C}_l$$

and

$$\mathcal{C}_{l-1} := \mathcal{C}_l \setminus \{C_l\}$$

is an optimal cycle packing of G_{l-1}. Iterating this argument, we obtain that for $i = l, (l-1), (l-2), \ldots, 2$, the ear P_i is contained in a unique cycle

$$C_i \in \mathcal{C}_i \subseteq \mathcal{C}_l$$

and that

$$\mathcal{C}_{i-1} := \mathcal{C}_l \setminus \{C_i, C_{i+1}, \ldots, C_l\}$$

is an optimal cycle packing of G_{i-1}. Note that this argument does not apply to $i = 1$, because $\nu(G_0) = \nu(G_1)$.

Since each of the ears in

$$\mathcal{E} = \{P_2, P_3, \ldots, P_l\}$$

is contained in a unique different cycle in $\mathcal{C}_l$, no internal vertex of any P_i is contained in any P_j for $2 \leq i \leq l$ and $1 \leq j \leq l$ with $i \neq j$. Since H is reduced and hence has no vertex of degree 2, this implies that the ears in $\mathcal{E}$ all have length 1, i.e. they are all edges.

Let

$$P = v_0 e_1 v_1 e_2 v_2 \ldots e_r v_r$$

be a maximal ear of G_1. Since G_1 is 2-connected and $k \geq 1$, the endvertices v_0 and v_r of P are of degree at least 3. Let

$$I = \{v_1, v_2, \ldots, v_{r-1}\}$$

be the set of internal vertices of P.

The next claim is obvious.

Claim A. *If an ear P_i for $2 \leq i \leq l$ has exactly one endvertex in I, then C_i contains either the edge e_1 or the edge e_r. Therefore, at most two ears in $\mathcal{E}$ have exactly one endvertex in I.*

Claim B. *No ear P_i for $2 \le i \le l$ has its two endvertices in I.*

Proof of Claim B. For contradiction, we assume that the index i with $2 \le i \le l$ is minimum such that P_i has the endvertices $v_x, v_y \in I$ for $1 \le x < y \le r-1$. Since $\nu(G_{i-1}) = \nu(G_i) - 1$, the cycle C_i is formed by P_i and the subpath P' of P between v_x and v_y. This implies that no internal vertex of P' is an endvertex of an ear $P_j \in \mathcal{E} \setminus \{P_i\}$. Hence P_i is an ear of H and C_i is a 1-cycle-path-subgraph of H.

Let H' arise from H by removing the ear P_i.

If $\nu(H') = \nu(H)$, we may choose $\tilde{G}_0 = H'$, $\tilde{P}_1 = P_i$ and $\tilde{G}_1 = H$ contradicting the choice of the sequence $G_0, G_1, \dots, G_l$ as shortest possible. Hence $\nu(H') = \nu(H) - 1$. This implies that H' has an optimal cycle packing not using the edges of P' and H is not reduced, which is a contradiction. □

Claim C. *G_1 does not contain a 2-cycle-path-subgraph.*

Proof of Claim C. For contradiction, we assume that Q is a 2-cycle-path-subgraph of G_1 with attachment vertices u and v. We may assume that $d_Q(u), d_Q(v) \ge 2$, i.e. that the 2 cycles C' and C'' of Q are the endblocks of Q.

Clearly, for every optimal cycle packing $\mathcal{C}'_1$ of G_1, we have $E_{C'} \cup E_{C''} \subseteq E_{\mathcal{C}'_1}$. This implies that $E_{C'} \cup E_{C''} \subseteq E_{\mathcal{C}_1}$ and, by Claims A and B, no ear in $\mathcal{E}$ has an endvertex in $V_Q \setminus \{u, v\}$. Hence Q is also a 2-cycle-path-subgraph of H and H is not reduced, which is a contradiction. □

Since G_1 arises by adding the ear P_1 to G_0, Claim C implies that G_0 does not contain any s-cycle-path-subgraph for $s \ge 6$. Since every s-cycle-path-subgraph for $s \le 5$ yields at most $2 \times 5 + 6 = 16$ maximal ears, this implies that the number of maximal ears of G_0 is at most $16|E_G|$ and hence the number of maximal ears of G_1 is at most $16|E_G| + 3$.

Since H is reduced and hence has no vertex of degree 2, Claim A implies that no maximal ear of G_1 has more than 2 internal vertices. This implies that the order $|V_{G_1}|$ and size $|E_{G_1}|$ of G_1 are bounded in terms of the size $|E_G|$ of G.

Since all ears in $\mathcal{E}$ are edges between vertices of G_1, the number of ears in $\mathcal{E}$ with different endvertices is bounded in terms of $|V_{G_1}|$, i.e. it is bounded in terms of $|E_G|$.

Furthermore, since all ears in $\mathcal{E}$ lie in different edge-disjoint cycles, the number of ears in $\mathcal{E}$ which have the same endvertices is bounded by the size $|E_{G_1}|$ of G_1, i.e. it is bounded in terms of $|E_G|$.

Altogether, G_1 arises from G by applying a subset of a set of operations whose cardinality is bounded in terms of $|E_G|$, and H arises from G_1 by applying a subset of a set of operations whose cardinality is also bounded in terms of $|E_G|$.

This completes the proof. □

The reader should note that the proof of Theorem 3 yields a – rather inefficient – algorithm which for $k \ge 1$ allows to derive $\mathcal{P}(k)$ from $\mathcal{P}(k-1)$ and has a running time which is bounded in terms of $|\mathcal{P}(k-1)|$ and the maximum size of graphs in $\mathcal{P}(k-1)$. Therefore, for every fixed k, we can – in principle – determine $\mathcal{P}(k)$ in finite time.

We finish this section with another algorithmic consequence of Theorem 3.

Let $k \in \mathbb{N}_0$ be fixed and let G be a fixed graph in $\mathcal{P}(k)$.

For a given 2-connected graph H as input, we can decide in polynomial time whether H extends G. The simplest argument implying this might be to consider all injective mappings of V_G to V_H and check whether the edges of G can be suitable replaced by cycle-paths in order to obtain H. This can clearly be done in polynomial time.

Therefore, in view of Lemma 1 and Theorem 3, for a given graph H as input, we can decide in polynomial time whether $\mu(H) - \nu(H) = k$. Furthermore, in view of the proof of Lemma 2, we can also efficiently construct an optimal cycle packing of H — even all of them — in this case.

3. $\mathcal{P}(1)$ and $\mathcal{P}(2)$

In this section we illustrate Theorem 3 and determine $\mathcal{P}(1)$ and $\mathcal{P}(2)$ explicitly.

The following lemma captures a straightforward yet important observation which was essentially also used in the proof of Theorem 3.

Lemma 4. *Let $k \ge 1$.*

(i) *Every graph $H \in \mathcal{P}(k)$ arises by adding an edge to a graph G such that either $\nu(G) = \nu(H)$ and G extends a graph in $\mathcal{P}(k-1)$, or $\nu(G) = \nu(H) - 1$ and G extends a graph in $\mathcal{P}(k)$.*

(ii) *Let $\mathcal{Q} \subseteq \mathcal{P}(k)$.*

If every graph H in $\mathcal{P}(k)$ which arises by adding an edge to a graph G such that either $\nu(G) = \nu(H)$ and G extends a graph in $\mathcal{P}(k-1)$, or $\nu(G) = \nu(H) - 1$ and G extends a graph in $\mathcal{Q}$, also belongs to $\mathcal{Q}$, then $\mathcal{Q} = \mathcal{P}(k)$.

Proof. (i) Let $H \in \mathcal{P}(k)$ and let P be the last ear in some ear decomposition of H.

Since H is reduced, P has length 1, i.e. it is an edge. Let G arise by removing P from H.

Clearly, $\mu(G) = \mu(H) - 1$ while $\nu(G) = \nu(H)$ or $\nu(G) = \nu(H) - 1$.

By the definition of $\mathcal{P}(k)$, $\nu(G) = \nu(H)$ implies that G extends a graph in $\mathcal{P}(k-1)$ and $\nu(G) = \nu(H) - 1$ implies that G extends a graph in $\mathcal{P}(k)$.

(ii) Let $H \in \mathcal{P}(k)$.

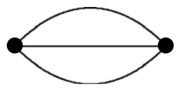

Fig. 2. $\mathcal{P}(1) = \{K_2^3\}$.

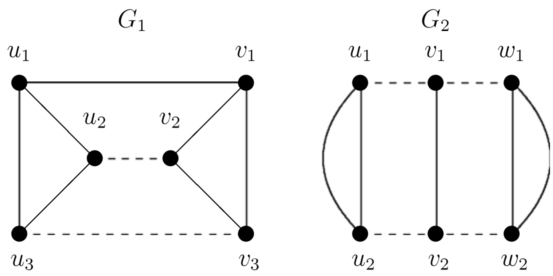

Fig. 3. The graphs $G_1, G_2 \in \mathcal{P}(2)$.

Iteratively deleting edges as in (i) and reducing the constructed graphs, we obtain a sequence $G_0, G_1, \dots, G_l$ such that $G_0 \in \mathcal{P}(k-1)$, $G_i \in \mathcal{P}(k)$ for $1 \le i \le l$, G_i contains an edge e_i such that $G_i - e_i$ extends G_{i-1} for $1 \le i \le l$ and $G_l = H$.

Since G_{i-1} has less edges than G_i for $1 \le i \le l$, the sequence is finite.

Inductively applying the hypothesis, we obtain that $G_i \in \mathcal{Q}$ for $1 \le i \le l$, i.e. $H \in \mathcal{Q}$ which implies $\mathcal{Q} = \mathcal{P}(k)$. $\square$

Note that Lemma 4 (ii) yields a criterion to check whether some subset $\mathcal{Q}$ of $\mathcal{P}(k)$ already contains all of $\mathcal{P}(k)$. Therefore, the proofs of the following two results reduce to tedious yet straightforward case analysis. The following result is in fact equivalent to a result in [5].

Theorem 5. *$\mathcal{P}(1) = \{K_2^3\}$ where K_2^3 is the unique graph with two vertices and three parallel edges (cf. Fig. 2).*

Proof. It is easy to verify that $K_2^3 \in \mathcal{P}(1)$.

Note that the only graphs extending graphs in $\mathcal{P}(0)$ are cycle-paths. This easily implies that, if $H \in \mathcal{P}(1)$ arises by adding an edge to a graph G with $\nu(G) = \nu(H)$ such that G extends a graph in $\mathcal{P}(0)$, then $H = K_2^3$.

Furthermore, if $H \in \mathcal{P}(1)$ arises by adding an edge to a graph G with $\nu(G) = \nu(H) - 1$ and G extends K_2^3, then H extends K_2^3. Since H is reduced, we obtain $H = K_2^3$.

By Lemma 4 (ii), the proof is complete. $\square$

We say that the graphs which arise from one of the two graphs G_1 or G_2 in Fig. 3 by contracting a subset of the edges indicated by dashed lines are *generated from* G_1 or G_2, respectively.

Theorem 6. *$\mathcal{P}(2)$ consists of K_4 and all graphs which are generated from G_1 or G_2.*

Proof. It is easy to verify that K_4 and all graphs which are generated from G_1 or G_2 belong to $\mathcal{P}(2)$.

Let $H \in \mathcal{P}(2)$.

We consider different cases.

Case 1. *H arises by adding an edge uv to a graph G with $\nu(G) = \nu(H) = 1$ such that G extends K_2^3.*

In this case G is a subdivision of K_2^3.

Since $\nu(H) = 1$, the vertices u and v are not contained in a common maximal ear of G. This implies that $H = K_4$.

Case 2. *H arises by adding an edge uv to a graph G with $\nu(G) = \nu(H) \ge 2$ such that G extends K_2^3.*

In this case G has a unique optimal cycle packing $\mathcal{C}$.

If $d_G(u) = d_G(v) = 2$ and u and v lie on a maximal ear contained in a cycle in $\mathcal{C}$, then $H = G_2$.

If $d_G(u) = d_G(v) = 2$ and u and v lie in different maximal ears contained in one cycle in $\mathcal{C}$, then H extends K_4. Since $H \ne K_4$, H is not reduced which is a contradiction.

If $d_G(u) = d_G(v) = 2$ and u and v lie in different cycles in $\mathcal{C}$, then H is generated from G_1.

If $d_G(u) \ge 3$, $d_G(v) = 2$ and v lies in a cycle in $\mathcal{C}$, then H extends K_4. Since $H \ne K_4$, H is not reduced which is a contradiction.

In all remaining subcases, H is generated from G_2.

Case 3. *H arises by adding an edge uv to a graph G with $\nu(G) = \nu(H) - 1$ such that G extends K_4.*

Let v_1, v_2, v_3, v_4 denote the vertices of K_4. We may assume that G arises by replacing the edges v_iv_j with $l_{i,j}$-cycle-paths $Q_{i,j}$.

Since H is reduced and $\nu(G) = \nu(H) - 1$, the vertices u and v are not both contained in one of the cycle-paths $Q_{i,j}$ and we obtain that H is generated from G_1.

Case 4. *H arises by adding an edge uv to a graph G with $\nu(G) = \nu(H) - 1$ such that G extends a graph generated from G_1.*

It is easy to verify that $\nu(G) = \nu(H) - 1$ implies that H is generated from G_1.

Case 5. *H arises by adding an edge uv to a graph G with $\nu(G) = \nu(H) - 1$ such that G extends a graph generated from G_2.*

It is easy to verify that $\nu(G) = \nu(H) - 1$ implies that H is generated from K_4 or G_2.

By Lemma 4 (ii), the proof is complete. $\square$

References

[1] B. Bollobás, Extremal graph theory, L. M. S. Monographs. 11. London, New York, San Francisco: Academic Press. XX, 488 p. (1978).
[2] A. Caprara, Sorting permutations by reversals and Eulerian cycle decompositions, SIAM J. Discrete Math. 12 (1999) 91–110.
[3] A. Caprara, A. Panconesi, R. Rizzi, Packing cycles in undirected graphs, J. Algorithms 48 (2003) 239–256.
[4] A. Caprara, R. Rizzi, Packing triangles in bounded degree graphs, Inform. Process. Lett. 84 (2002) 175–180.
[5] J. Degenhardt, P. Recht, On a relation between the cycle packing number and the cyclomatic number of a graph, manuscript (2008).
[6] P. Erdős, L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen 9 (1962) 3–12.
[7] M. Krivelevich, Z. Nutov, M.R. Salavatipour, J. Yuster, R. Yuster, Approximation algorithms and hardness results for cycle packing problems, ACM Trans. Algorithms 3 (Article No. 48) (2007).
[8] J.W. Moon, On edge-disjoint cycles in a graph, Canad. Math. Bull. 7 (1964) 519–523.
[9] D. Rautenbach, F. Regen, On packing shortest cycles in graphs, Inform. Process. Lett. 109 (2009) 816–821.
[10] A. Schrijver, Combinatorial Optimization Polyhedra and Efficiency, Springer-Verlag, Berlin, Heidelberg, 2004.
[11] M. Simonovits, A new proof and generalizations of a theorem of Erdős and Posa on graphs without $k + 1$ independent circuits, Acta Math. Acad. Sci. Hung. 18 (1967) 191–206.
[12] H.-J. Voss, Über die Taillenweite in Graphen, die genau k knotenunabhängige Kreise enthalten, und über die Anzahl der Knotenpunkte, die in solchen Graphen alle Kreise repräsentieren, Dissertationsschrift TH Ilmenau (1966).
[13] H. Walther, H.-J. Voss, Über Kreise in Graphen, Berlin: VEB Deutscher Verlag der Wissenschaften. 271 S. m. 99 Abb. (1974).
[14] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932) 339–362.